Organization

Part 1: Constraint Systems

Part 2: Widening and Narrowing

Widening and Narrowing

Observation 1

Cousot, Cousot 1977

- Program invariants are post-solutions of systems of equations over suitable lattices.

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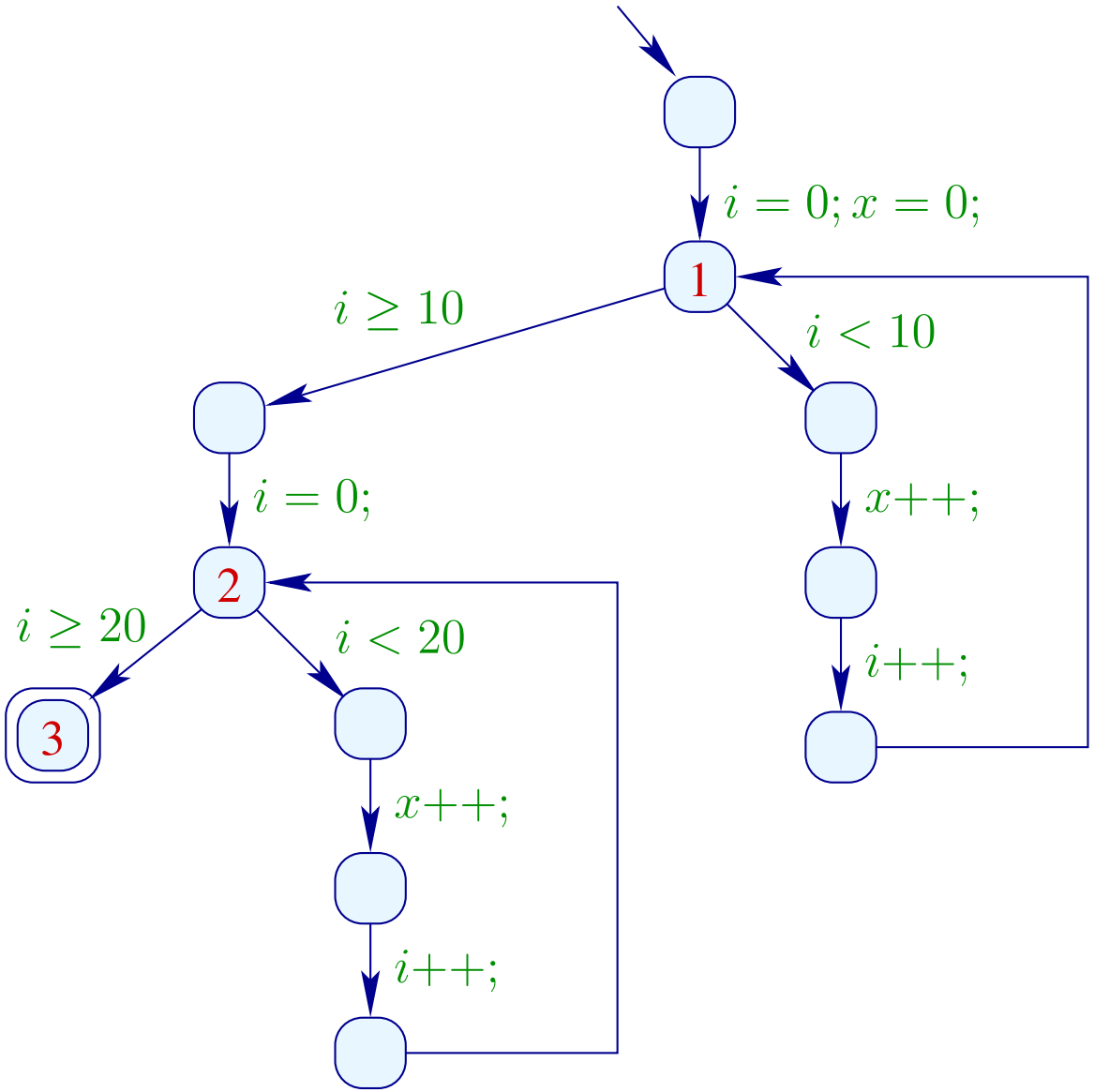
- Some of the precision subsequently can may be recovered through narrowing ...


$i = 0; x = 0;$
1
$i \geq 10$
$i < 10$
$i = 0;$
$x{+}{+};$
2
$i \geq 20$
$i < 20$
$i{+}{+};$
3
$x{+}{+};$
$i{+}{+};$

Example Analysis

reduced product of

- intervals
- linear equalities

$\Longrightarrow$ infinite ascending chains

$\Longrightarrow$ widening/narrowing required

Widening

Equations

$$x_i = f_i(x_1, \ldots, x_n) \ , \quad i = 1, \ldots, n$$

Widening

Equations

$$x_i = x_i \sqcup f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n$$

Widening

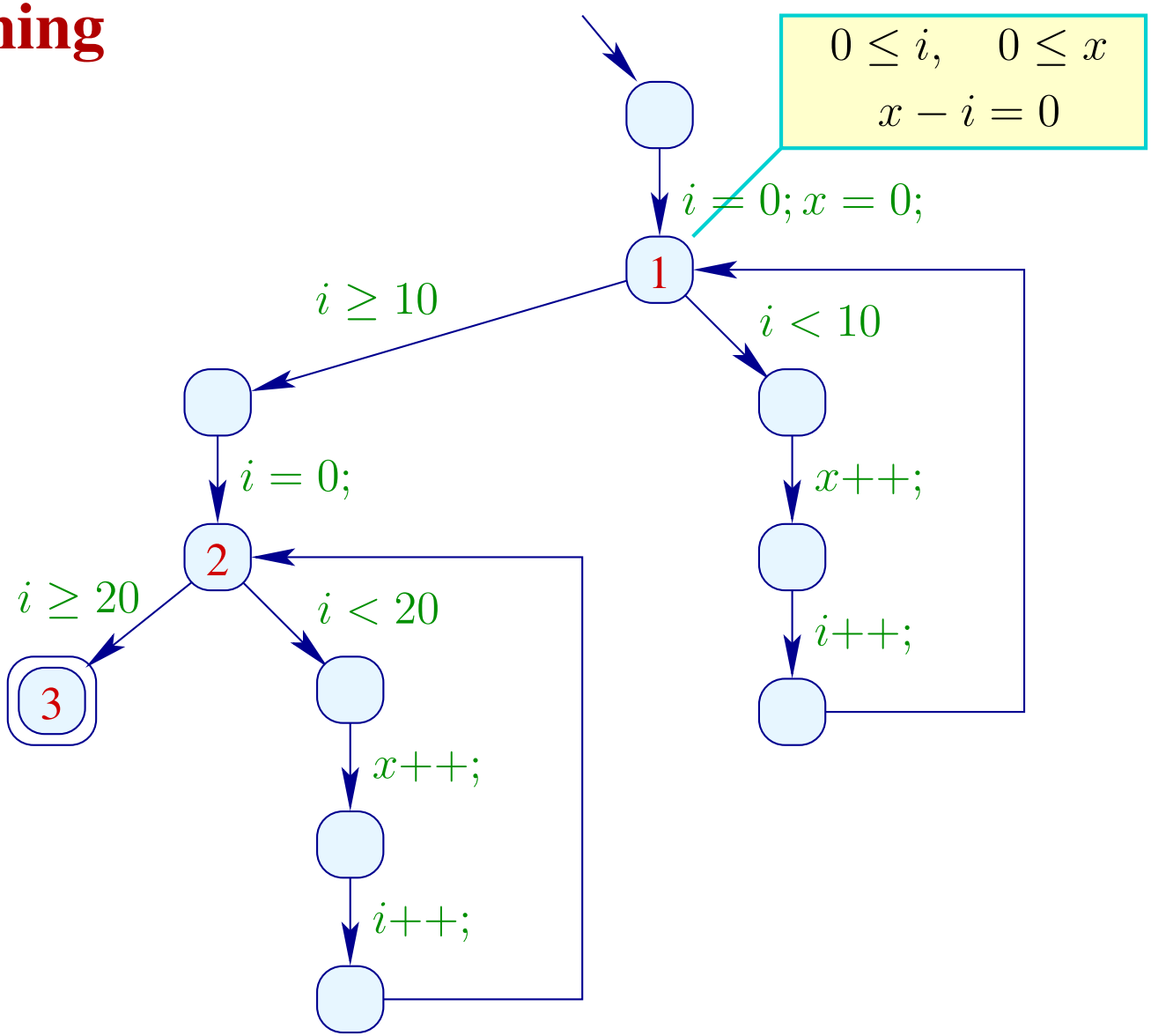
Equations

$$x_i = x_i \uplus f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n$$

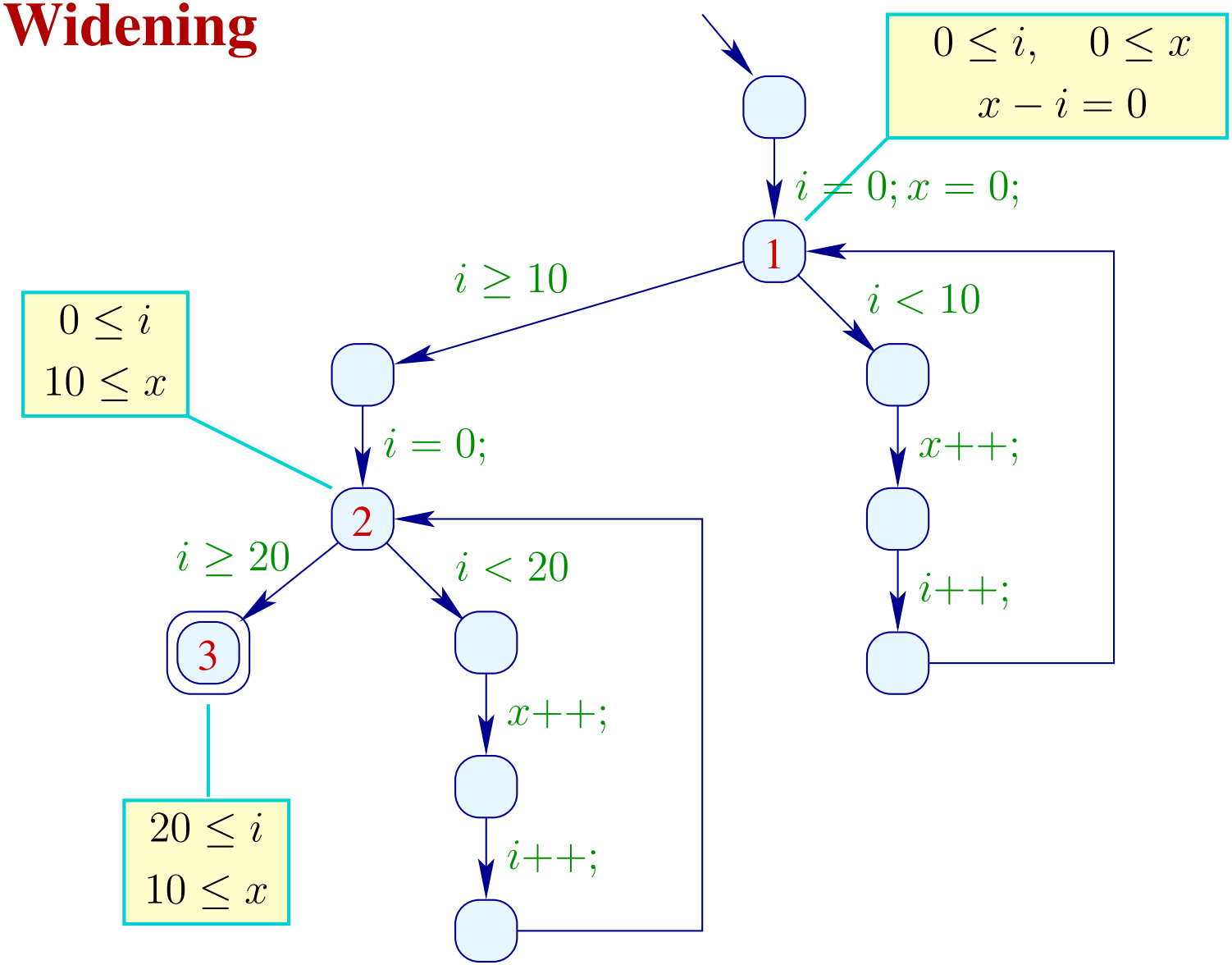
Widening operator

$$a \sqcup b \sqsubseteq a \uplus b$$

Widening



Widening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Narrowing

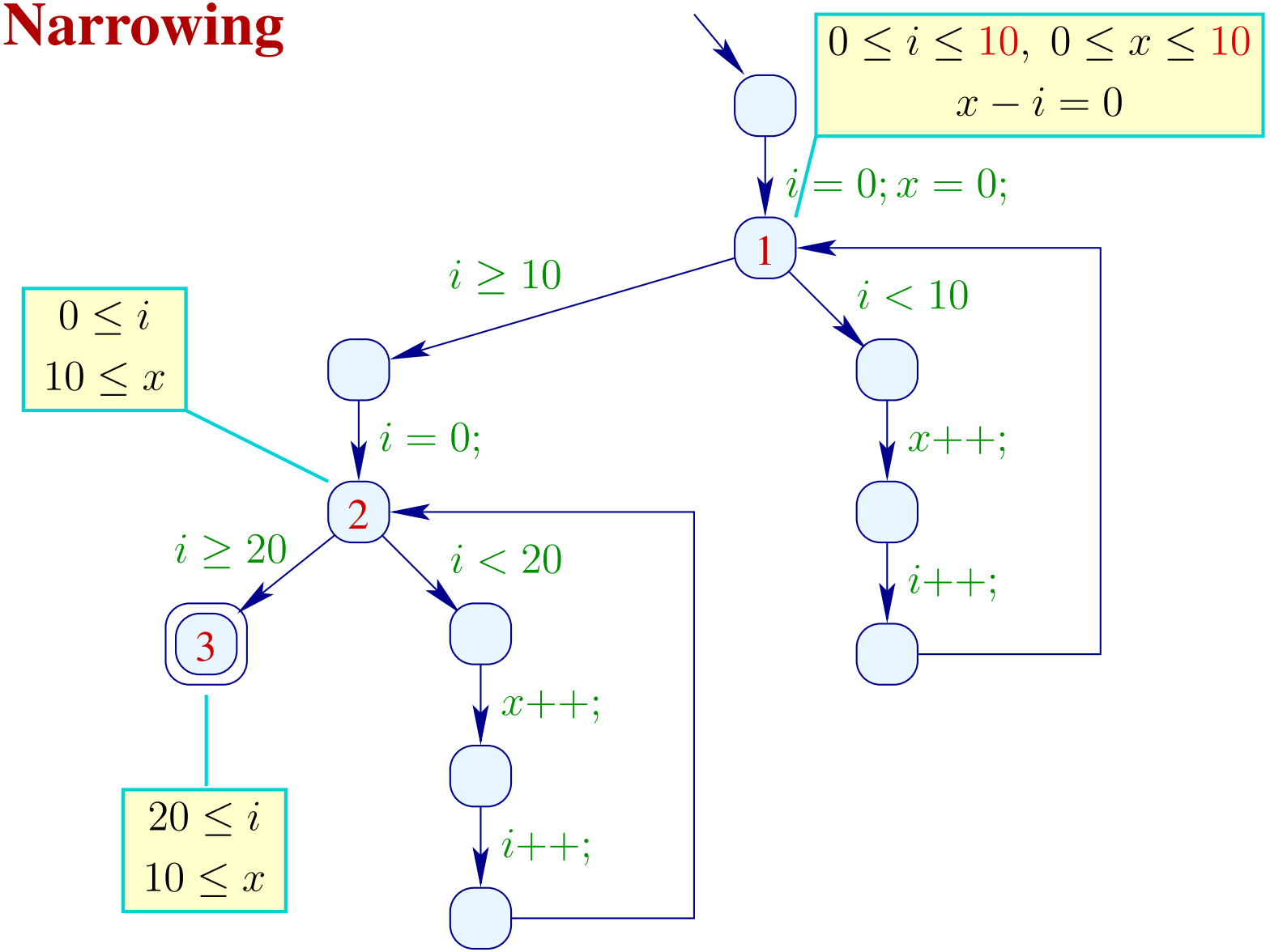
Equations

$$x_i = x_i \sqcap f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n$$

Narrowing operator

$$b \sqsubseteq a \quad \Longrightarrow \quad b \sqsubseteq a \sqcap b \sqsubseteq a$$

Narrowing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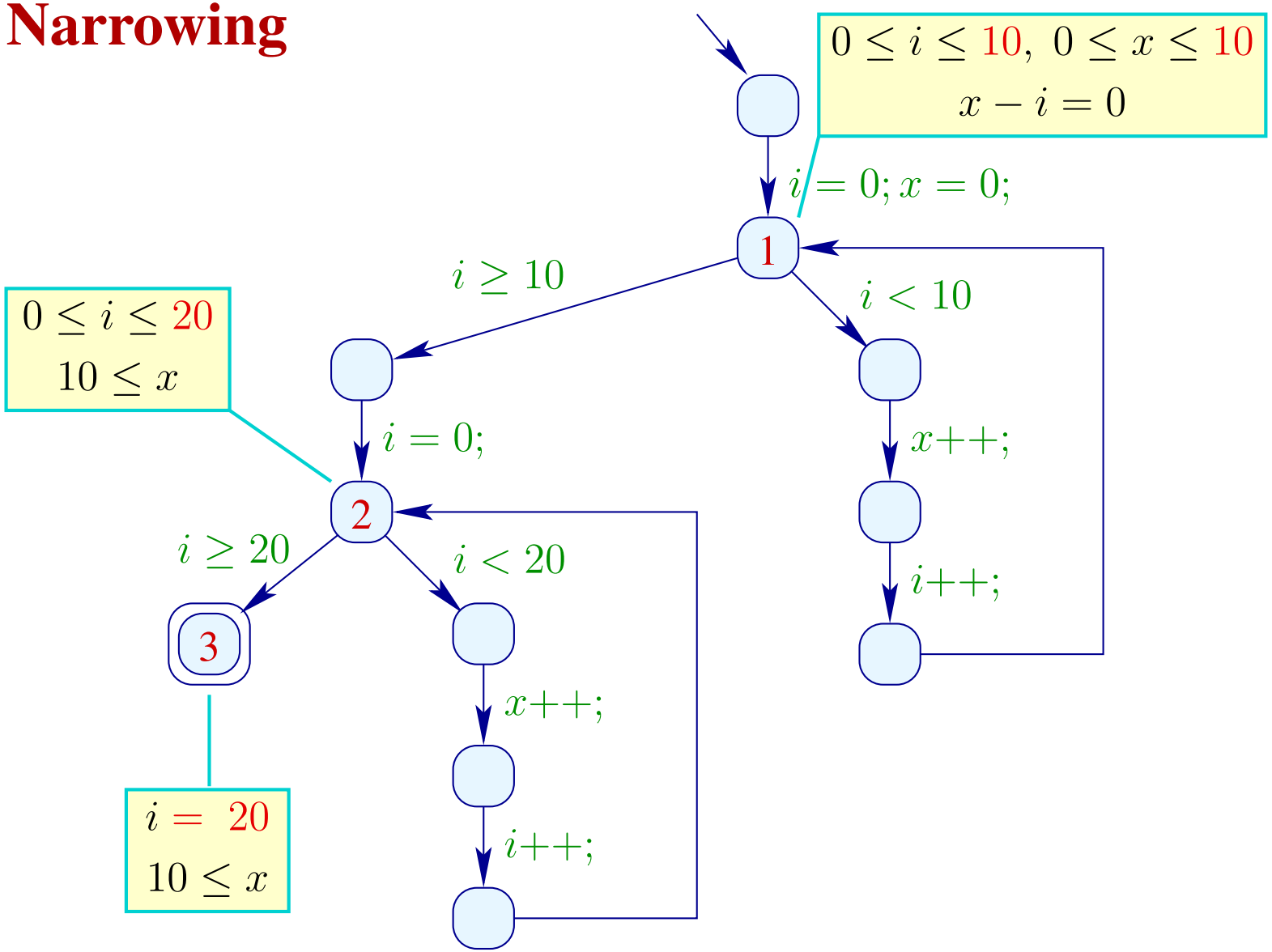
Narrowing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Problem

- Precision, once abandoned, is difficult to recover.
- Narrowing is only partially defined and requires:
  - ... equations
  - ... monotonic right-hand sides

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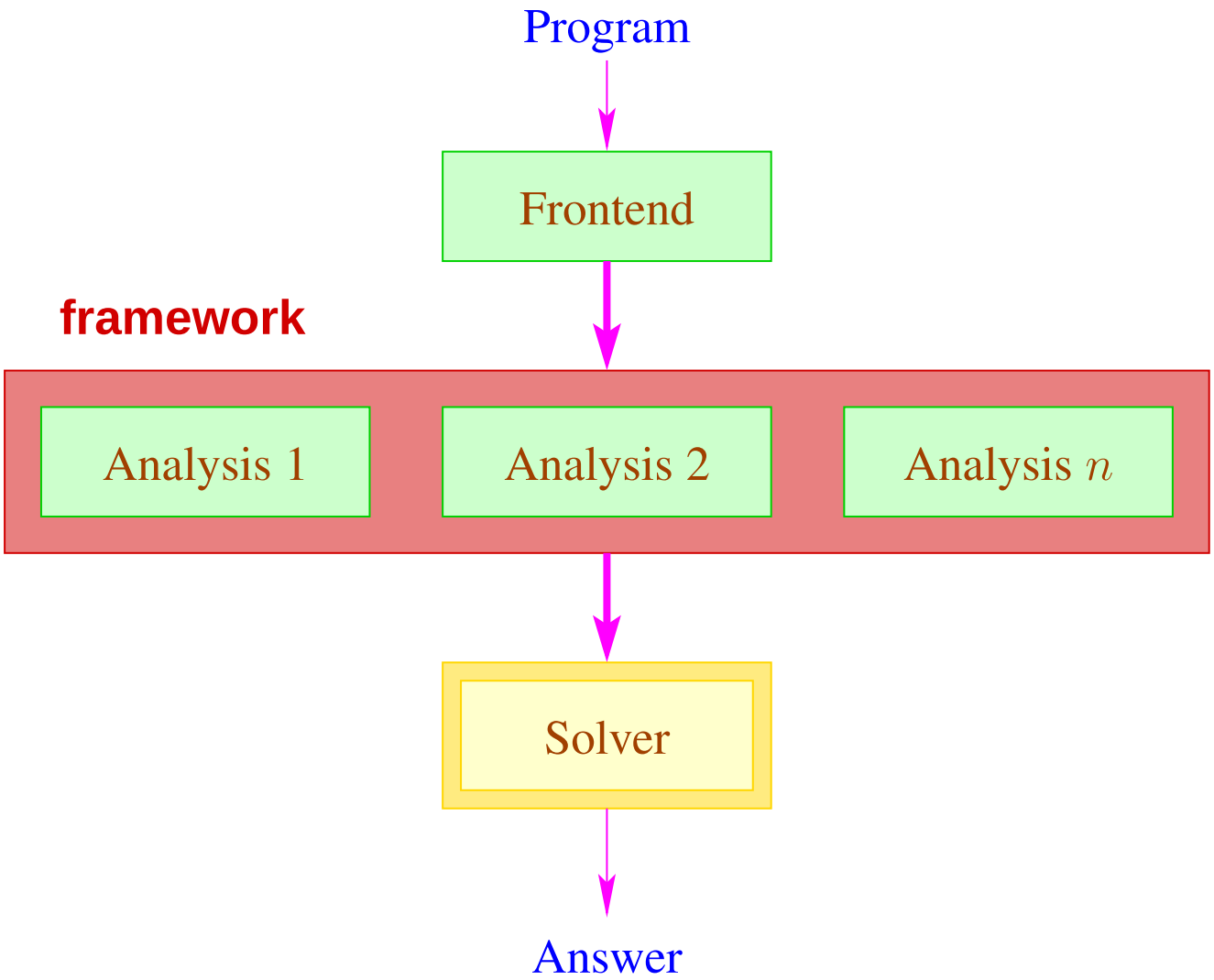
- Narrowing is only partially defined and requires:

  ... equations

  ... monotonic right-hand sides

  $\Longrightarrow$ not met by interprocedural analysis ...

Goblint:


Program
Frontend
framework
Analysis 1
Analysis 2
Analysis n
Solver
Answer

Observation 2

Analysis	Constraint system
no context	finite monotonic finite dependences
context	infinite non-monotonic finite dependences
partial context	infinite non-monotonic infinite dependences

Observation 2

Analysis	Constraint system
no context	finite monotonic finite dependences
context	infinite non-monotonic finite dependences
partial context	infinite non-monotonic infinite dependences

... 2-phase algorithm is unsound !

Overview

- Combining widening and narrowing
- Termination
- Evaluation
- Enhancements

Combining Widening and Narrrowing

Equations

$$x_i = x_i \boxplus f_i(x_1, \ldots, x_n) \ , \quad i = 1, \ldots, n$$

Combining Widening and Narrrowing

Equations

$$x_i = x_i \boxplus f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n$$

Combined operator

$$a \boxplus b = \begin{array}{l} \textbf{if } b \sqsubseteq a \textbf{ then } a \boxplus b \\ \textbf{else } a \sqcup b \end{array}$$

Idea

- Perform one joint iteration !
- Iterate until stabilization !

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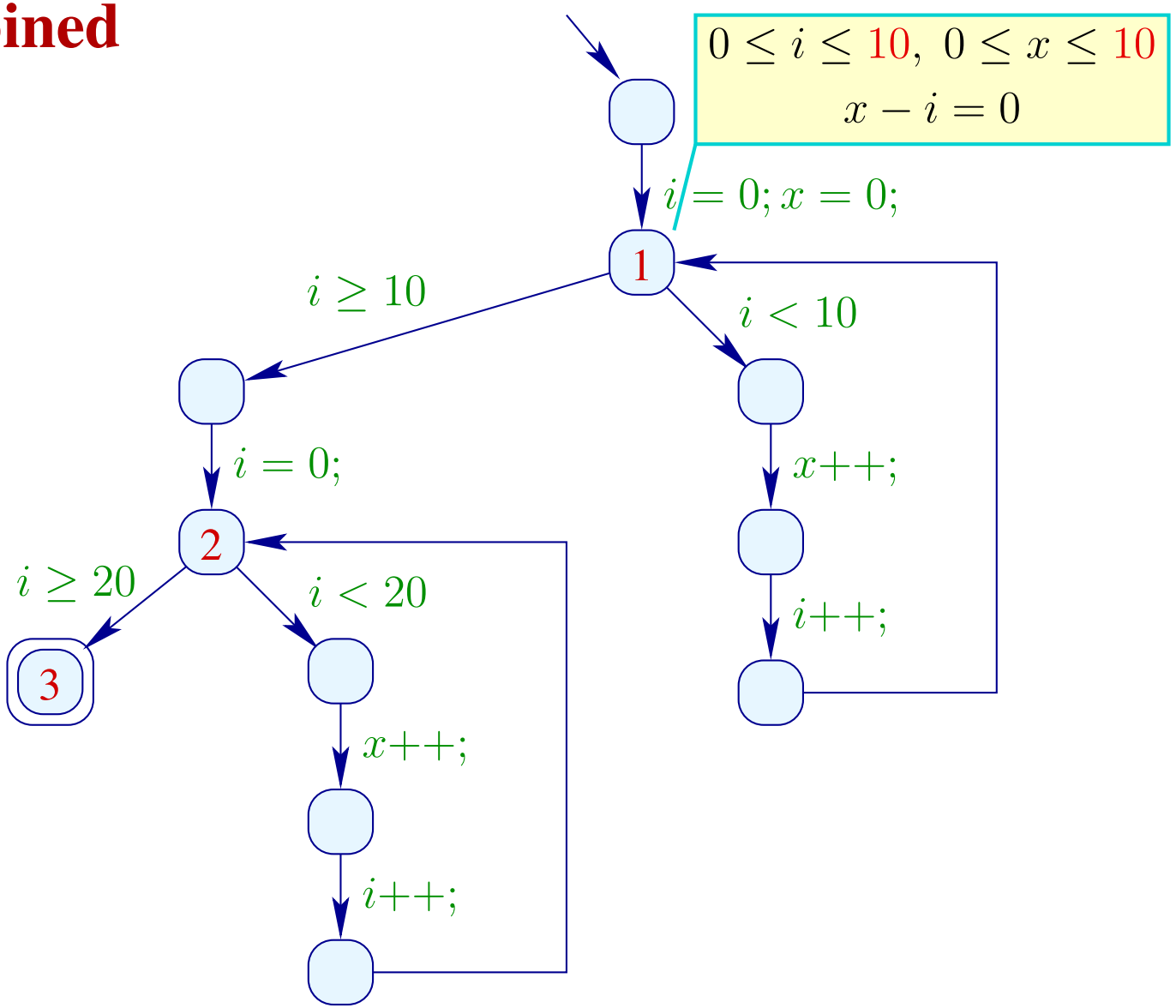
Theorem

Assume that the fixpoint algorithm performs a sequence of atomic evaluations of right-hand sides. Then

- Upon termination, a post-solution is attained ...
- This holds independent of the start values !

Combined



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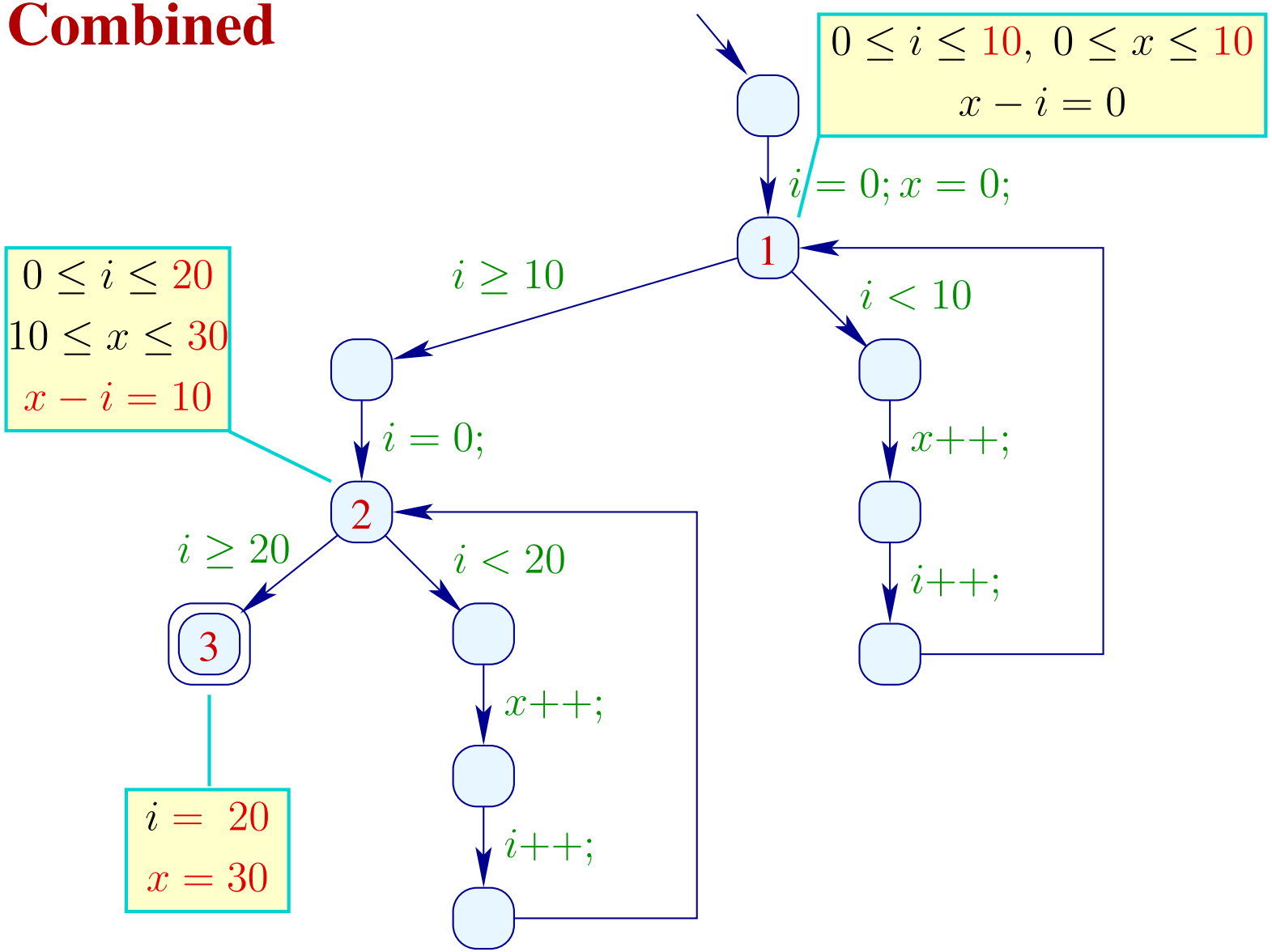
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Termination

- Any solver performing chaotic fixpoint iteration can be enhanced to a ⊟-solver.
- Termination, though, can no longer be guaranteed—even if right-hand sides are monotonic ...

Round robin iteration

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 + 1 \\ x_3 &= x_1 \end{aligned}$$

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for $\mathbb{N} \cup \{\infty\}$ with $\leq$ results in:

	0	1	2	3	4	5	6
x_1	0	0	∞	1	∞	2	$\ldots$
x_2	0	∞	1	∞	2	∞	$\ldots$
x_3	0	0	∞	1	∞	2	$\ldots$

Work list iteration

$$
\begin{aligned}
x_1 &= x_1 + 1 \wedge x_2 + 1 \\
x_2 &= x_2 + 1 \wedge x_1 + 1
\end{aligned}
$$

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results in:

	1	2	3	4	5	6	7	8	
W	$[x_1, x_2]$	$[x_1, x_2]$	$[x_1, x_2]$	$[x_2]$	$[x_2, x_1]$	$[x_2, x_1]$	$[x_1]$	$[x_1, x_2]$	
x_1	0	∞	1	1	1	1	1	∞	...
x_2	0	0	0	0	∞	2	2	2	...

Bad News

Standard solvers fail to terminate already for trivial systems of equations.

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Good News

Variations of standard solvers are guaranteed to terminate—whenever right-hand sides are monotonic ...

Structured RR

```
void solve i {
    if (i = 0) return;
    solve (i − 1);
    new ← ρ(x_i) ⊟ f_i ρ;
    if (ρ(x_i) ≠ new) {
        ρ(x_i) ← new;
        solve i;
    }
}
```

Structured RR (cont.)

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 + 1 \\ x_3 &= x_1 \end{aligned}$$

results in:

	0	1	2	3	4	5	6	7
x_1	0	0	∞	∞	1	1	1	∞
x_2	0	∞	∞	1	1	1	∞	∞
x_3	0	0	0	0	0	∞	∞	∞

// SRR is a mild generalization of iterating

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Theorem

(1) If SRR terminates, a post-solution has been found.

(2) SRR terminates, whenever all right-hand sides are monotonic.

Complexity Issues

- If $\sqcup\!\!\!\!-\!\!\!\!\sqcup = \sqcup$ and $\sqcap\!\!\!\!-\!\!\!\!\sqcap = \pi_1$, then SRR may perform in the worst-case, only half as many iterations as RR.

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- For non-monotonic systems, it still may not terminate

  // which we also have not observed.

Structured W

$Q \leftarrow \emptyset$; **for** $(i \leftarrow 1; i \leq n; i{+}{+})$ add Q x_i;
while $(Q \neq \emptyset)$ {
  $x_i \leftarrow$ extract_min (Q);
  $new \leftarrow \rho(x_i)$ ▤ $f_i\,\rho$;
  if $(\rho(x_i) \neq new)$ {
    $\rho(x_i) \leftarrow new$; add Q x_i;
    forall $(x_j \in \mathsf{infl}_i)$ add Q x_j;
  }
}

Structured W

$Q \leftarrow \emptyset$; **for** $(i \leftarrow 1; i \leq n; i{+}{+})$ add Q x_i;

while $(Q \neq \emptyset)$ { // Q priority queue

  $x_i \leftarrow$ extract_min (Q);

  $new \leftarrow \rho(x_i)$ ▤ $f_i\,\rho$;

  if $(\rho(x_i) \neq new)$ {

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Structured W (cont.)

$$\begin{aligned} x_1 &= x_1 + 1 \wedge x_2 + 1 \\ x_2 &= x_2 + 1 \wedge x_1 + 1 \end{aligned}$$

results in:

	1	2	3	4	5	6	7	8
Q	$[x_1, x_2]$	$[x_1, x_2]$	$[x_1, x_2]$	$[x_2]$	$[x_1, x_2]$	$[x_1, x_2]$	$[x_2]$	[]
x_1	0	∞	1	1	1	∞	∞	∞
x_2	0	0	0	0	∞	∞	∞	∞

Theorem

(1) If SW terminates, a post-solution has been found.

(2) SW terminates, whenever all right-hand sides are monotonic.

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- For arbitrary W/N operators and monotonic systems, it may perform exponentially many evaluations

  // although we never observed that in practice
- For non-monotonic systems, it still may not terminate

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Local Solving

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- Partial contexts and the combination with flow-insensitive analysis can conveniently be handled using side-effecting ...

Local Solving (cont.)

- Two-phase widening/narrowing neither works well with local solving nor with side-effecting.

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$\Longrightarrow$ new ⊟-solver SLR^+

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Local solver SLR$^+$ does not perform any preprocessing of system of equations.

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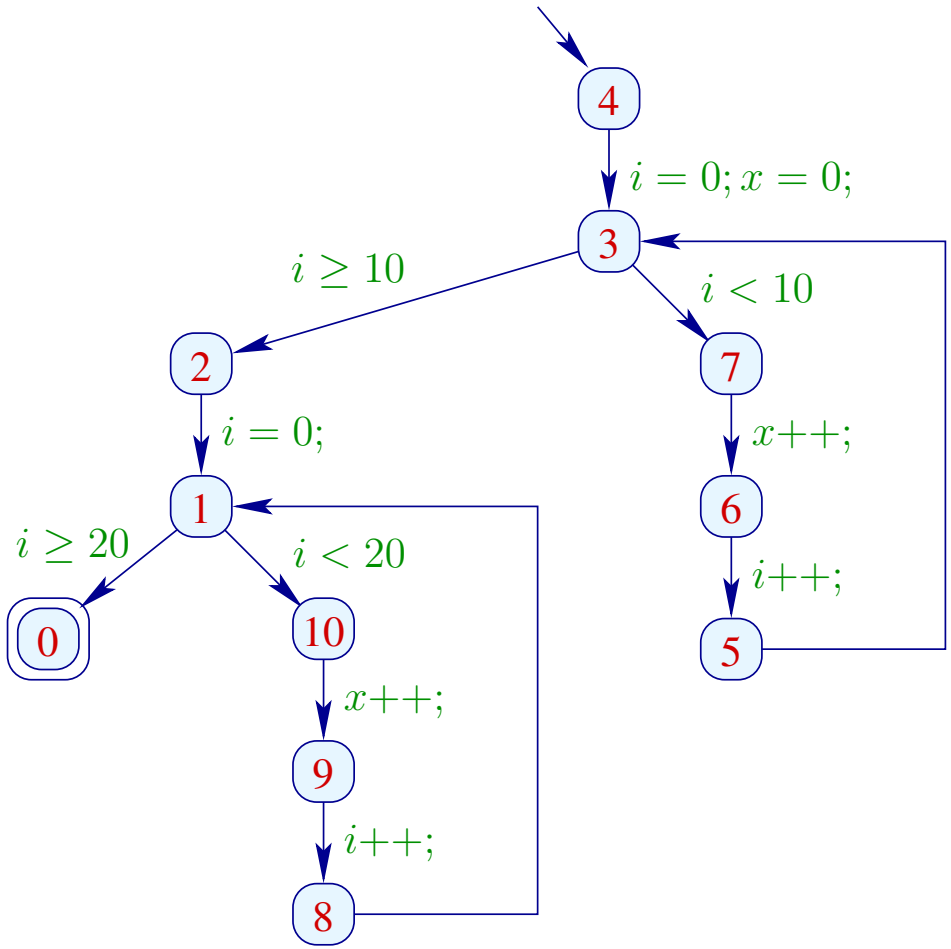
Idea

Exploit priorities ...

Assignment of priorities starts at end point of the program and then recursively descends into variable dependences ...



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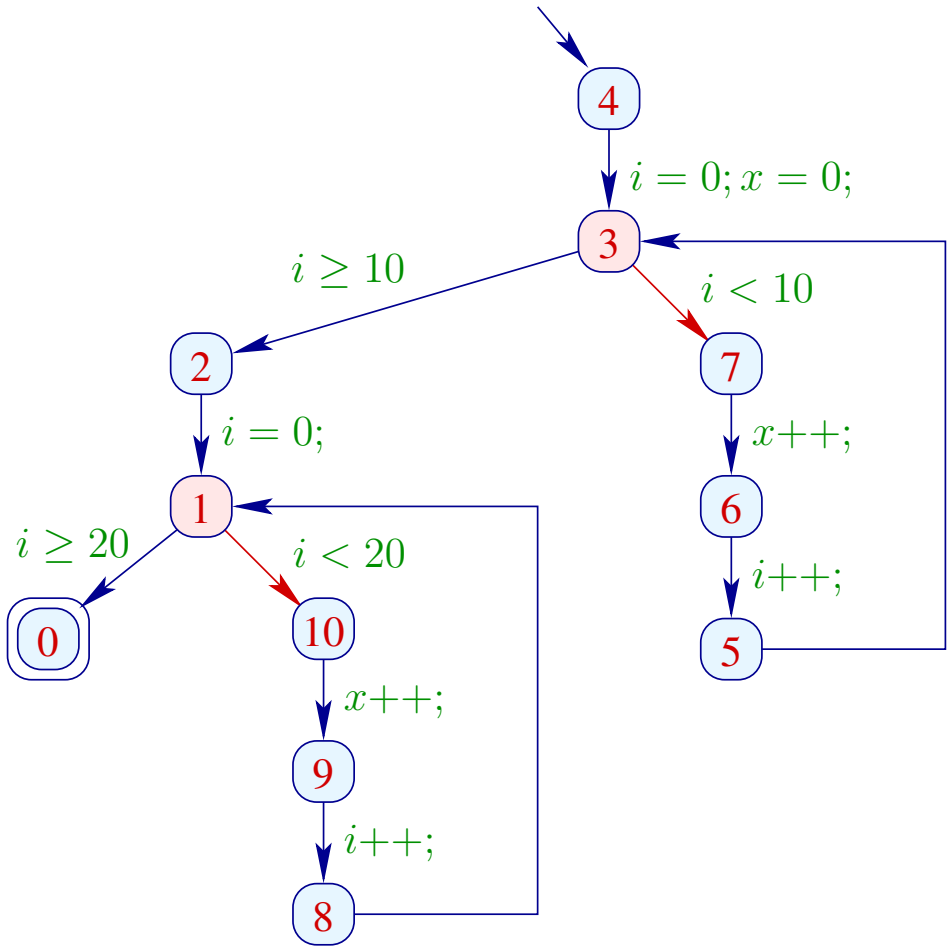


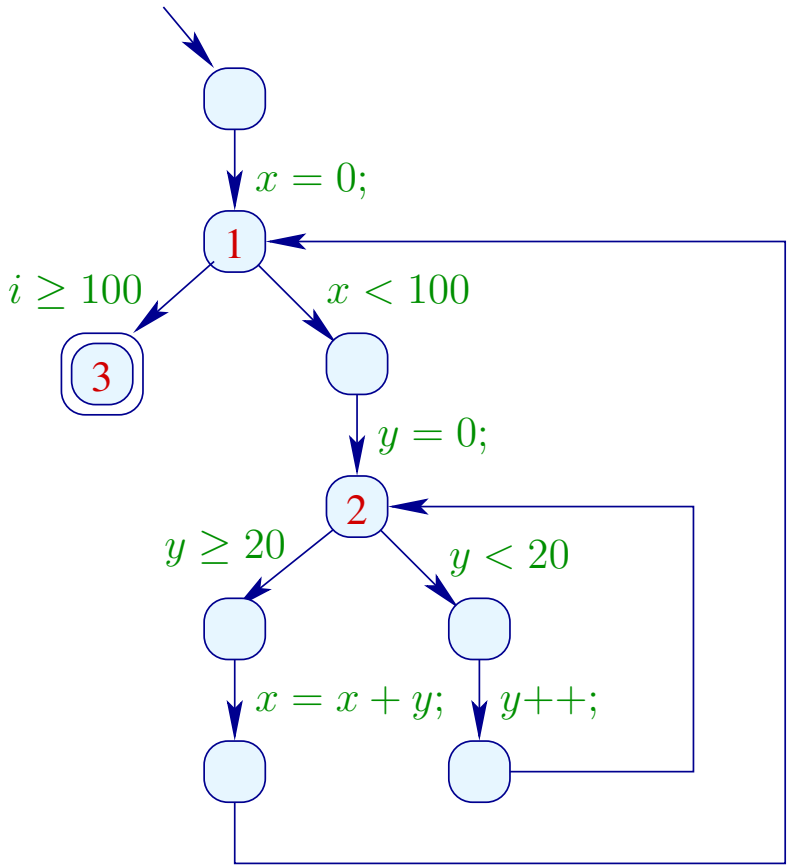
Theorem

(1) If SLR^+ terminates, a post-solution has been found.

(2) SLR^+ terminates, whenever all right-hand sides are monotonic and only finitely many unknowns are encountered.

Back Edges

Combined W/N still may be doomed when loops are nested ...



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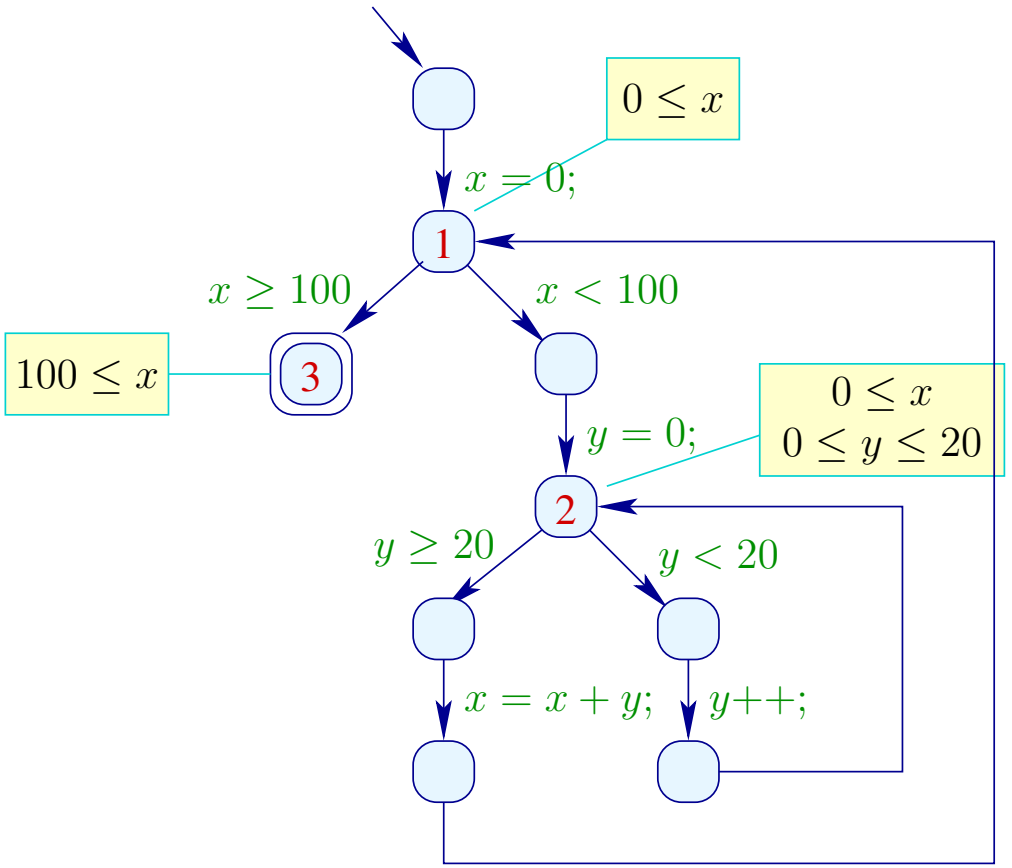
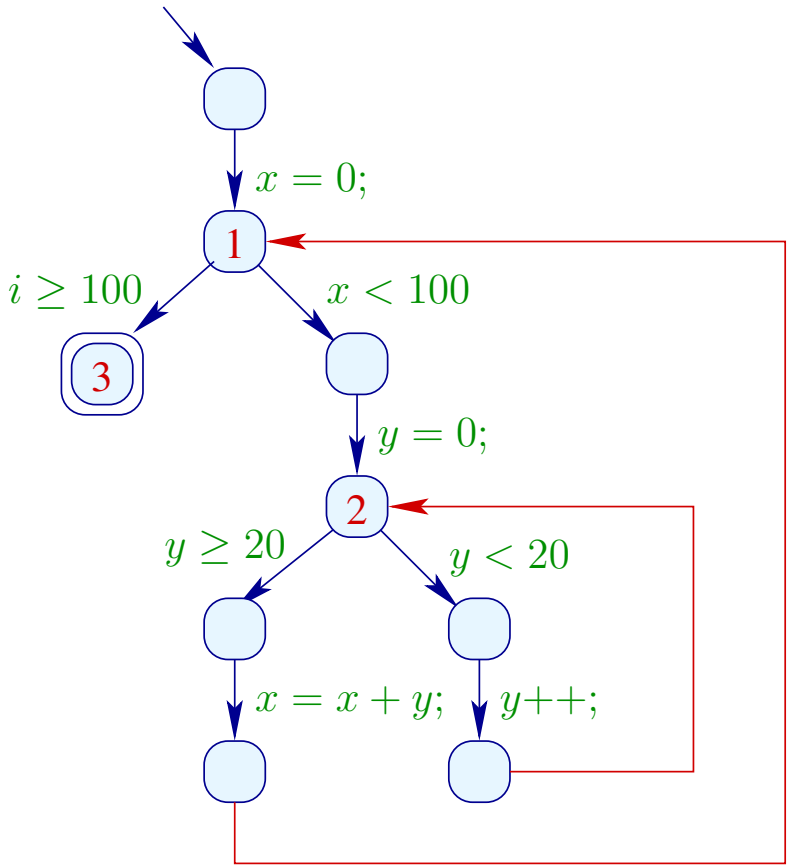


Restrict W/N to back edges: Amato, Scozzari 2013

$$z = z \boxplus (\mathsf{outside} \sqcup \mathsf{back})$$



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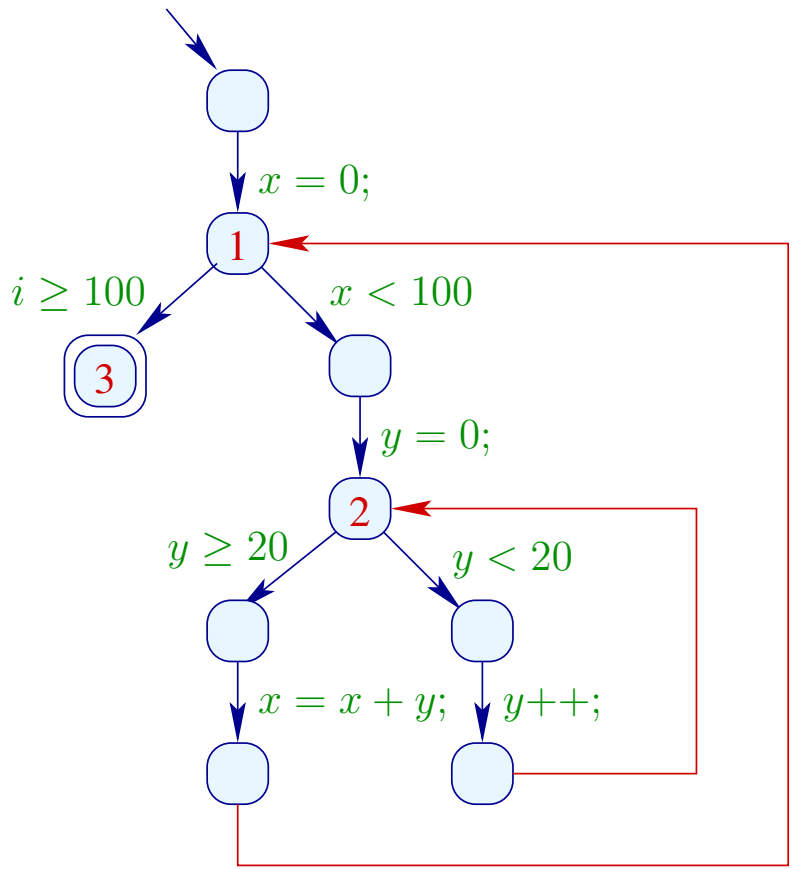

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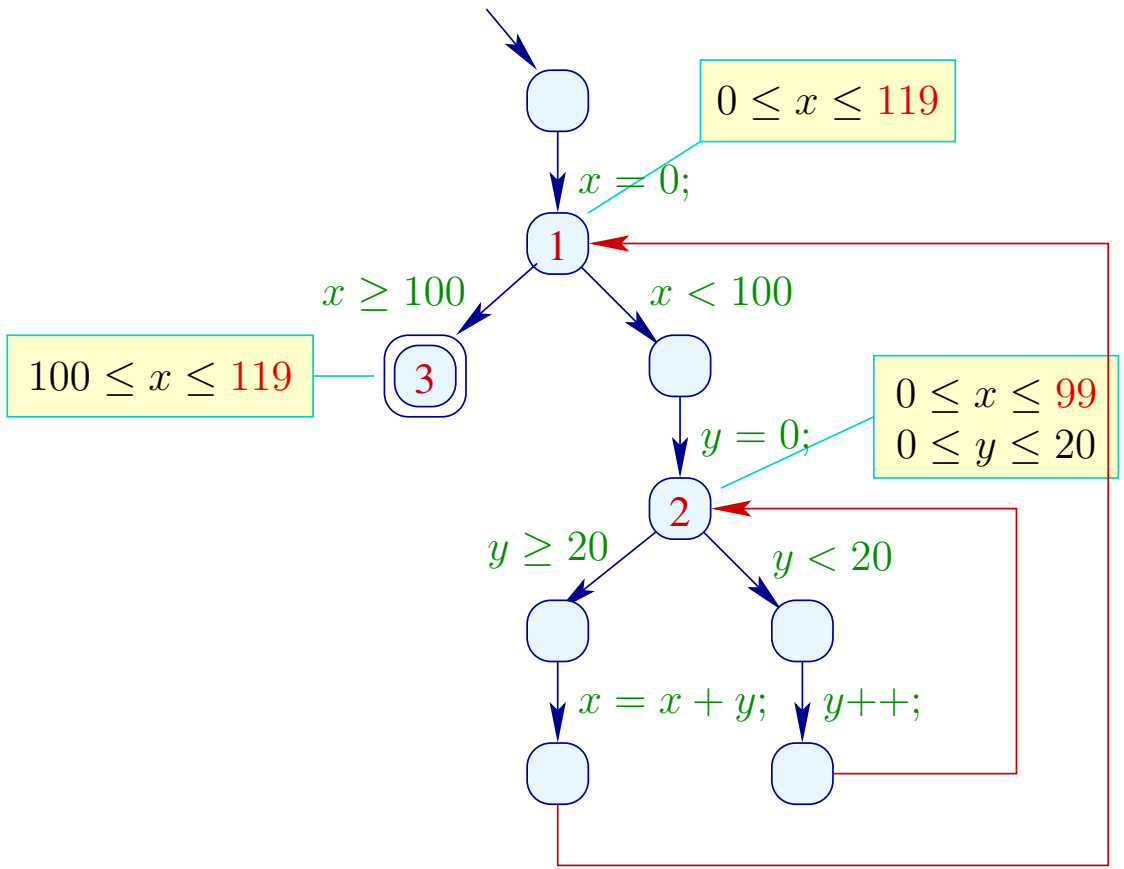

Discussion

- Restricting $\exists$ to back edges, may increase precision significantly !
- It can be easily integrated into SRR, SW without compromising termination !!

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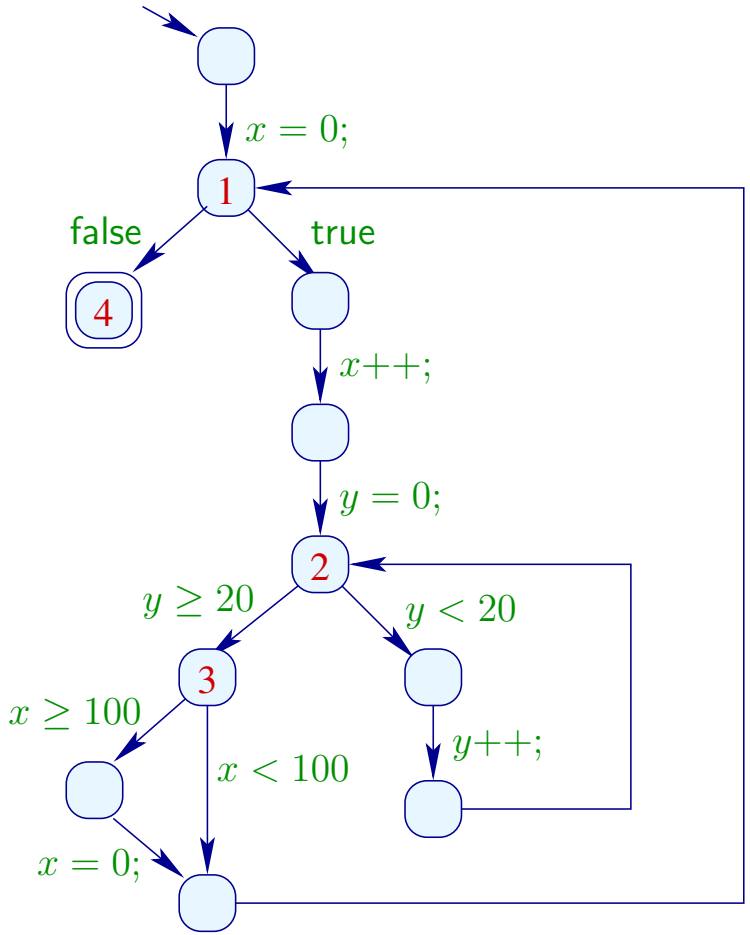
- Restricting $\boxminus$ to back edges, may increase precision significantly !
- It can be easily integrated into SRR, SW without compromising termination !!
- It is not yet clear how it can be integrated into local solvers such as SLR^+ where no preprocessing is possible.

Restart

Information loss in inner loops can be quite insistent ...



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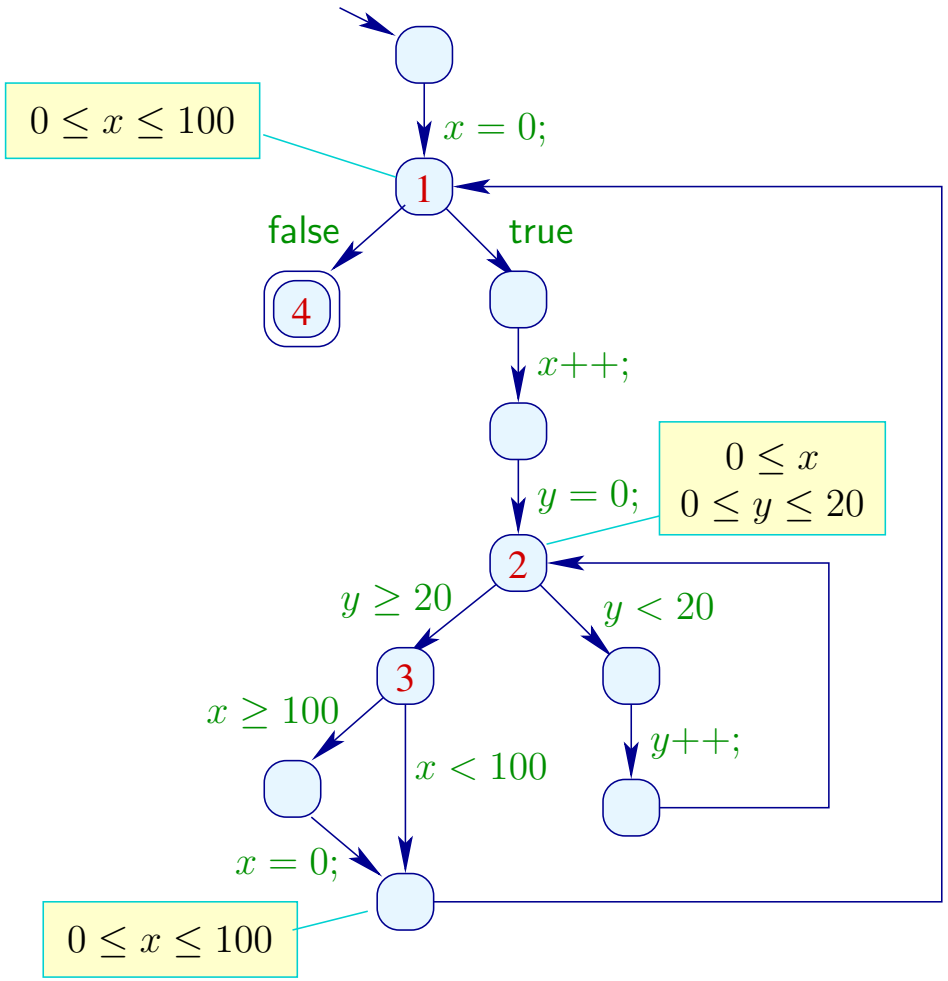
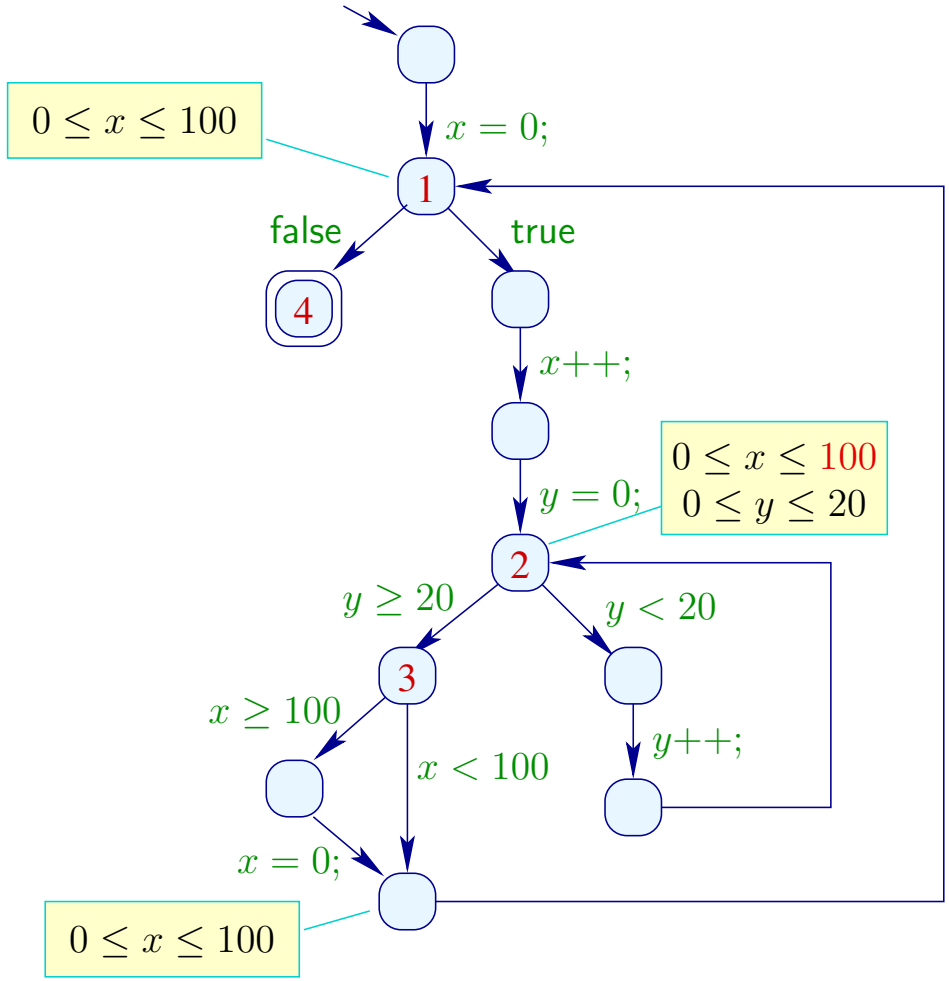


Amato, Scozzari 2013

At decrease, restart iteration on lower-priority unknowns ...



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Discussion

- After restarting of lower-priority unknowns, the value of x may be further decreased !
- Restarting can be integrated into any structured iterator without compromising termination !!
- In theory, it may drastically increase complexity.
- In practice, it seems to cause only a slowdown by a small factor.

Experiments

Precision 2-phase vs. 目-solving

Efficiency 目-solving against 凵-solving

Experiments

Precision 2-phase vs. ⊟-solving

benchmark suite of the Märdalen WCET research group

Efficiency ⊟-solving against ⊔-solving

C projects of the SpecCpu2006 benchmark suite that can be handled by CIL

Results

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- A significant of amount of precision is retained by combining widening and narrowing !
- Back-edge aware iteration as well restarting may each provide further (small) improvements in precision.
- In absence of context, the $\boxplus$-solver is only marginally slower than the corresponding $\sqcup$-solver.
- Restarting is not completely impractical: in our experiments, the penalty was just a small factor !!

Summary

- A decent fixpoint engine should support lattices with infinite ascending chains.

  A generic idea which works for any lattice is W/N.

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  A generic idea which works for any lattice is W/N.

- The ⊟-operator can be plugged into any reasonable solver — at the price of potential non-termination.

- Termination can be enforced for monotonic systems when finitely many unknowns are encountered—by modifying the fixpoint algorithms appropiately.

- These seem to terminate practically — also for our nonmonotonic systems !

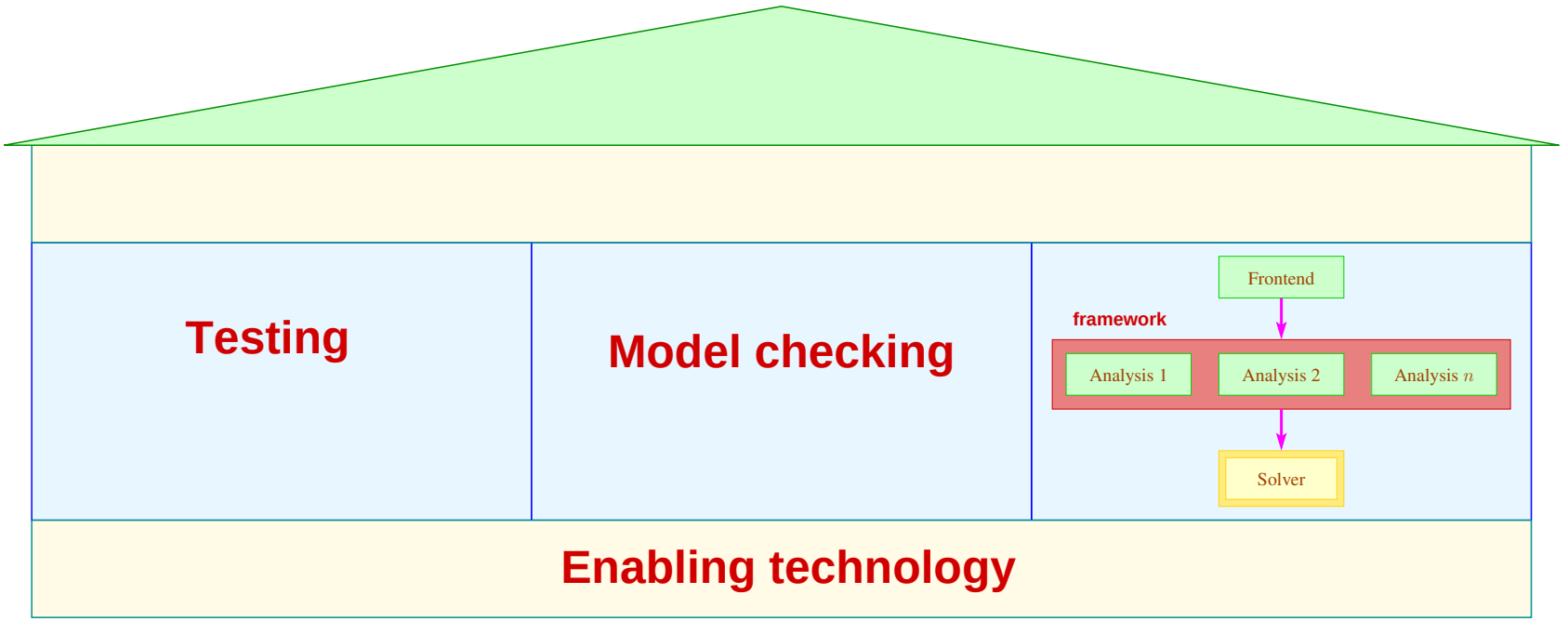
Summary (cont.)

- In order to achieve maximal precision, the solver can
  - apply w/n reluctantly
  - e.g., only at back edges
  - restart solving when values start decreasing

Summary (cont.)

- In order to achieve maximal precision, the solver can
  - apply w/n reluctantly
  - e.g., only at back edges
  - restart solving when values start decreasing
- The extra precision may come at an extra cost.
- ... which is surprisingly small !

Dream


Testing
Model checking
framework
Frontend
Analysis 1
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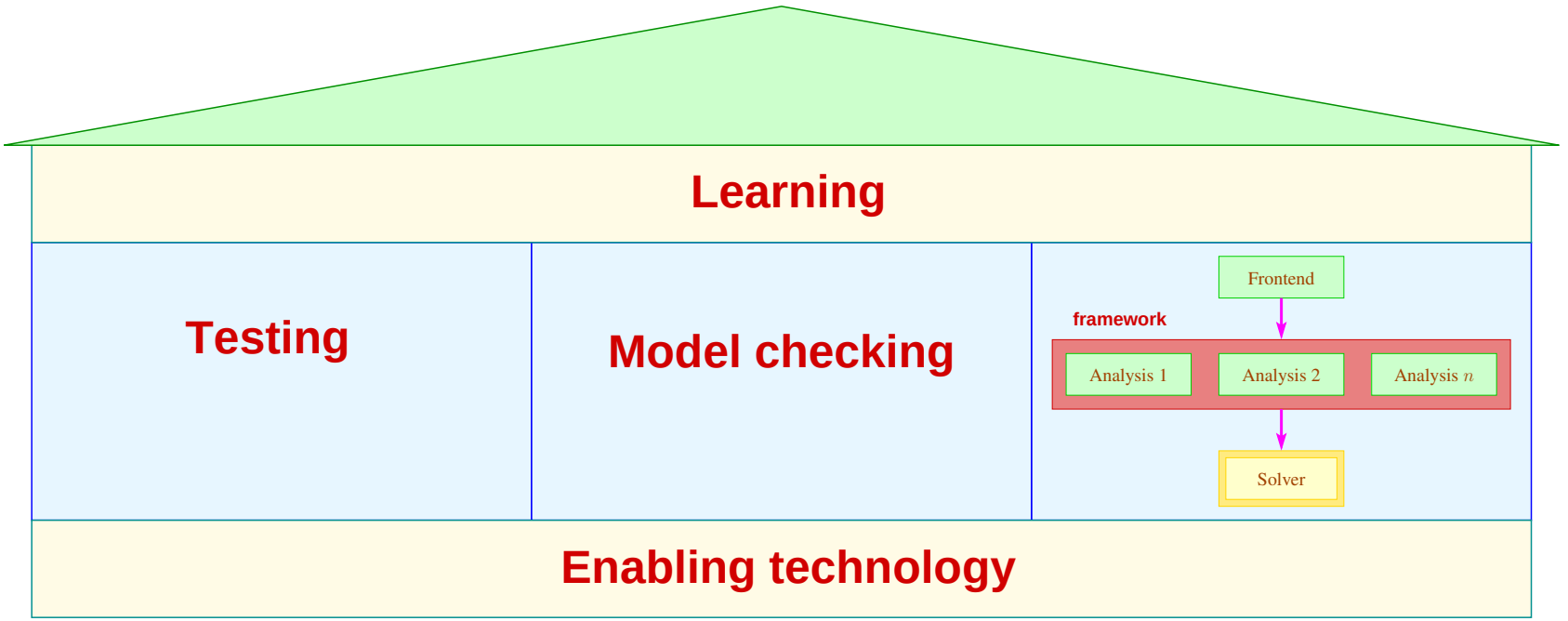
Solver
Enabling technology

Dream


Learning
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